Final Exam — Functional Analysis (WIFA–08)

Monday 9 April 2018, 18.30h–21.30h

University of Groningen

Instructions

1. The use of calculators, books, or notes is not allowed.
2. All answers need to be accompanied with an explanation or a calculation: only answering “yes”, “no”, or “42” is not sufficient.
3. If p is the number of marks then the exam grade is $G = 1 + p/10$.

Problem 1 (6 + 10 + 3 + 3 + 3 = 25 points)

Assume $0 < w_k \leq 1$ for all $k \in \mathbb{N}$ and define the following normed linear space:

$$\mathcal{W} = \left\{ x = (x_1, x_2, x_3, \dots) \,:\, x_k \in \mathbb{K},\ \sup_{k\in\mathbb{N}} |x_k| w_k < \infty \right\}, \qquad \|x\|_{\mathcal{W}} = \sup_{k\in\mathbb{N}} |x_k| w_k.$$

(a) Prove that $\|\cdot\|_{\mathcal{W}}$ is a norm on $\mathcal{W}$.

(b) Prove that $(\mathcal{W}, \|\cdot\|_{\mathcal{W}})$ is a Banach space.

(c) Recall the following Banach space from the lecture notes:

$$\ell^\infty = \left\{ x = (x_1, x_2, x_3, \dots) \,:\, x_k \in \mathbb{K},\ \sup_{k\in\mathbb{N}} |x_k| < \infty \right\}, \qquad \|x\|_\infty = \sup_{k\in\mathbb{N}} |x_k|.$$

Show that $\ell^\infty \subset \mathcal{W}$ and $\|x\|_{\mathcal{W}} \leq \|x\|_\infty$ for all $x \in \ell^\infty$.

(d) Assume $\inf_{k\in\mathbb{N}} w_k > 0$. Prove that $\ell^\infty = \mathcal{W}$ and that the norms $\|\cdot\|_\infty$ and $\|\cdot\|_{\mathcal{W}}$ are equivalent.

(e) Assume $w_k = 1/k$ for all $k \in \mathbb{N}$. Show that the inclusion $\ell^\infty \subset \mathcal{W}$ is strict and that the norms $\|\cdot\|_\infty$ and $\|\cdot\|_{\mathcal{W}}$ are *not* equivalent on ℓ^∞.

Problem 2 (4 + 6 + 5 + 5 + 5 = 25 points)

Let $(X, \|\cdot\|)$ be an infinite-dimensional Banach space over $\mathbb{K}$ and let $T \in B(X)$ be of the form

$$Tx = f(x)z,$$

where $f \in X' = B(X, \mathbb{K})$ and $z \in X$ are fixed and nontrivial.

(a) Compute $\|T\|$.

(b) Determine all eigenvalues of T.

(c) Show that $T^n = f(z)^{n-1}T$ for all $n \in \mathbb{N}$.

Turn page for parts (d) and (e).

(d) Assume that $|\lambda| > \|T\|$. Show that

$$(T-\lambda)^{-1} = -\frac{1}{\lambda} - \frac{1}{\lambda(\lambda - f(z))} T.$$

(e) Determine $\rho(T)$ and hence $\sigma(T)$.

Problem 3 (5 + 3 + 7 + 5 = 20 points)

(a) Formulate the uniform boundedness principle.

Let X and Y be Banach spaces over $\mathbb{K}$ and let $Q : X \times Y \to \mathbb{K}$ be a bilinear map, i.e. linear in the first entry and linear in the second entry. Assume that:

(i) $\forall\, x \in X \;\; \exists\, M_x \geq 0$ such that $|Q(x,y)| \leq M_x \|y\| \;\forall\, y \in Y$;

(ii) $\forall\, y \in Y \;\; \exists\, N_y \geq 0$ such that $|Q(x,y)| \leq N_y \|x\| \;\forall\, x \in X$.

Prove the following statements:

(b) For each nonzero $y \in Y$ the functional $T_y \in L(X, \mathbb{K})$ defined by $T_y(x) = Q(x,y)/\|y\|$ is bounded;

(c) $\sup_{y \neq 0} \|T_y\| < \infty$;

(d) There exists $K \geq 0$ such that $|Q(x,y)| \leq K \|x\| \|y\|$ for all $x \in X$ and $y \in Y$.

Problem 4 (3 + 7 + 4 + 6 = 20 points)

(a) Formulate the Hahn-Banach Theorem for normed linear spaces.

Let X be a normed linear space, and assume that $V \subset X$ is a finite-dimensional linear subspace. Let $\{e_1, \ldots, e_n\}$ be a basis for V.

(b) Show that for each $i = 1, \ldots, n$ there exists $f_i \in X'$ such that $f_i(e_j) = \delta_{ij}$.

(c) Prove that the linear map

$$P : X \to X, \quad Px = \sum_{i=1}^{n} f_i(x) e_i.$$

is a projection and bounded.

(d) Prove the following properties:

(i) $\ker P$ and $\operatorname{ran} P$ are closed;

(ii) $\ker P \cap \operatorname{ran} P = \{0\}$;

(iii) $X = \ker P + \operatorname{ran} P$.

End of test (90 points)

Solution of problem 1 (6 + 10 + 3 + 3 + 3 = 25 points)

(a) Clearly, $\|x\|_{\mathcal{W}} \geq 0$ for all $x \in \mathcal{W}$. If $\|x\|_{\mathcal{W}} = 0$, then $|x_k|w_k = 0$ for all $k \in \mathbb{N}$. Since $w_k > 0$ it follows that $x_k = 0$ for all $x \in \mathcal{W}$, which means that $x = 0$.
(2 points)

If $\lambda \in \mathbb{K}$ and $x \in \mathcal{W}$, then

$$\|\lambda x\| = \sup_{k\in\mathbb{N}} |\lambda x_k|w_k = \sup_{k\in\mathbb{N}} |\lambda||x_k|w_k = |\lambda| \sup_{k\in\mathbb{N}} |x_k|w_k = |\lambda|\, \|x\|_{\mathcal{W}}.$$

(2 points)

If $x, y \in \mathcal{W}$, then

$$\begin{aligned}\|x+y\|_{\mathcal{W}} &= \sup_{k\in\mathbb{N}} |x_k + y_k|w_k \\ &\leq \sup_{k\in\mathbb{N}} (|x_k| + |y_k|)w_k \\ &\leq \sup_{k\in\mathbb{N}} |x_k|w_k + \sup_{k\in\mathbb{N}} |x_k|w_k = \|x\|_{\mathcal{W}} + \|y\|_{\mathcal{W}}.\end{aligned}$$

(2 points)

(b) *Solution 1.* Let x^n be a Cauchy sequence in $\mathcal{W}$. Let $\epsilon > 0$ be arbitrary, then there exists $N \in \mathbb{N}$ such that

$$m, n \geq N \quad \Rightarrow \quad \|x^n - x^m\|_{\mathcal{W}} \leq \epsilon,$$

or, equivalently,

$$m, n \geq N \quad \Rightarrow \quad |x_k^n - x_k^m|w_k \leq \epsilon \quad \text{for all } k \in \mathbb{N}. \tag{1}$$

This shows that for fixed $k \in \mathbb{N}$ the sequence (x_k^n) is a Cauchy sequence in $\mathbb{K}$.
(2 points)

Since $\mathbb{K}$ is complete there exists $x_k \in \mathbb{K}$ such that $x_k^n \to x_k$ as $n \to \infty$. Now we define $x = (x_1, x_2, x_3, \dots)$ and show that $x \in \mathcal{W}$ and $\|x^n - x\|_{\mathcal{W}} \to 0$ as $n \to \infty$.
(2 points)

Letting $m \to \infty$ in equation (1) and using the fact that inequalities are preserved under taking limits gives

$$n \geq N \quad \Rightarrow \quad |x_k^n - x_k|w_k \leq \epsilon \quad \text{for all } k \in \mathbb{N},$$

or, equivalently,

$$n \geq N \quad \Rightarrow \quad \|x^n - x\|_{\mathcal{W}} \leq \epsilon, \tag{2}$$

which indeed shows that $x^n \to x$ in $\mathcal{W}$.
(3 points)

Note that equation (2) shows that $x^N - x \in \mathcal{W}$. Since $x^N \in \mathcal{W}$ by assumption and the fact that $\mathcal{W}$ is a linear space it follows that $x = x^N - (x^N - x) \in \mathcal{W}$ as desired.
(3 points)

Solution 2. Define the following linear map

$$T : \mathcal{W} \to \ell^\infty, \quad (x_1, x_2, x_3, \dots) \mapsto (x_1 w_1, x_2 w_2, x_3 w_3, \dots).$$

It is clear that T is bijective and isometric, i.e., $\|Tx\|_\infty = \|x\|_{\mathcal{W}}$.
(3 points)

If x^n is a Cauchy sequence in $\mathcal{W}$, then Tx^n is a Cauchy sequence in ℓ^∞. Since ℓ^∞ is complete there exists $y \in \ell^\infty$ such that $Tx^n \to y$. Since T^{-1} is also isometric, and in particular bounded, it follows that $x^n \to T^{-1}y$.
(7 points)

(c) Let $x = (x_1, x_2, x_3, \dots) \in \ell^\infty$. Since $0 < w_k \leq 1$ for al $k \in \mathbb{N}$ it follows that $|x_k| w_k \leq |x_k|$ for all $k \in \mathbb{N}$ which implies that

$$\|x\|_{\mathcal{W}} = \sup_{k \in \mathbb{N}} |x_k| w_k \leq \sup_{k \in \mathbb{N}} |x_k| = \|x\|_\infty < \infty,$$

which also implies that $x \in \mathcal{W}$.
(3 points)

(d) Let $c := \inf_{k \in \mathbb{N}} w_k > 0$ and $x = (x_1, x_2, x_3, \dots)$. We have $c \leq w_k$ for all $k \in \mathbb{N}$ so that $c|x_k| \leq |x_k| w_k$ for all $k \in \mathbb{N}$. Taking the supremum gives the inequality $c\|x\|_\infty \leq \|x\|_{\mathcal{W}}$. Together with part (c) this shows that the norms $\|\cdot\|_{\mathcal{W}}$ and $\|\cdot\|_\infty$ are equivalent. This also shows that $\|x\|_\infty < \infty$ whenever $\|x\|_{\mathcal{W}} < \infty$ so that $\mathcal{W} = \ell^\infty$.
(3 points)

(e) Take $x = (1, 2, 3, 4, \dots)$, then clearly $\|x\|_{\mathcal{W}} = 1$ so that $x \in \mathcal{W}$. However, $x \notin \ell^\infty$, which means that the inclusion $\mathcal{W} \subset \ell^\infty$ is strict.
(2 points)

Now take $x^n = (1, 2, \dots, n, 0, 0, \dots)$. Clearly, $x^n \in \mathcal{W} \cap \ell^\infty$ and

$$\frac{\|x^n\|_\infty}{\|x^n\|_{\mathcal{W}}} = n \to \infty.$$

Hence, there exists no constant $c > 0$ such that $c\|x\|_\infty \leq \|x\|_{\mathcal{W}}$ for all $x \in \ell^\infty$.
(1 point)

Solution of problem 2 (4 + 6 + 5 + 5 + 5 = 25 points)

(a) We have

$$\|T\| = \sup_{x\neq 0} \frac{\|Tx\|}{\|x\|} = \sup_{x\neq 0} \frac{|f(x)|\|z\|}{\|x\|} = \|z\| \sup_{x\neq 0} \frac{|f(x)|}{\|x\|} = \|f\|\|z\|.$$

(4 points)

(b) Since X is infinite-dimensional we can find $w \in X$ such that z and w are linearly independent. If $f(z) = 0$, then $z \in \ker T$. If $f(w) = 0$, then $w \in \ker T$. If both $f(z) \neq 0$ and $f(w) \neq 0$, then $x_0 = f(w)z - f(z)w \neq 0$ (since z and w are linearly independent) and $x_0 \in \ker T$. This proves that $0 \in \sigma_p(T)$.
(3 points)

Assume that $Tx = \lambda x$ for some nontrivial $x \in X$, then $f(x)z = \lambda z$. Therefore, $x = cz$ for some constant $c \neq 0$. This gives

$$f(cz)z = \lambda cz \quad \Rightarrow \quad f(z)z = \lambda z \quad \Rightarrow \quad \lambda = f(z),$$

which means that $f(z)$ is an eigenvalue of T. Hence, $\sigma_p(T) = \{0, f(z)\}$.
(3 points)

(c) For $n = 1$ the statement is obvious. Assume that the statement is true for some $n \in \mathbb{N}$, then

$$\begin{aligned} T^{n+1}x &= T^n(Tx) \\ &= f(z)^{n-1}T^2x \\ &= f(z)^{n-1}T(f(x)z) \\ &= f(z)^{n-1}f(x)Tz \\ &= f(z)^{n-1}f(x)f(z)z \\ &= f(z)^nTx, \end{aligned}$$

which shows that the statement is also true for $n+1$. By induction the assertion is true for all $n \in \mathbb{N}$.
(5 points)

(d) Assume that $|\lambda| > \|T\|$ so that $\|T/\lambda\| < 1$. The geometric series gives

$$\begin{aligned} (T-\lambda)^{-1} &= -\frac{1}{\lambda}\left(I - \frac{T}{\lambda}\right)^{-1} \\ &= -\frac{1}{\lambda}\sum_{n=0}^{\infty} \frac{T^n}{\lambda^n} \\ &= -\frac{1}{\lambda} - \frac{1}{\lambda}\left(\sum_{n=1}^{\infty} \frac{f(z)^{n-1}}{\lambda^n}\right)T \\ &= -\frac{1}{\lambda} - \frac{1}{\lambda^2}\left(\sum_{n=0}^{\infty} \frac{f(z)^n}{\lambda^n}\right)T \\ &= -\frac{1}{\lambda} - \frac{1}{\lambda(\lambda - f(z))}T. \end{aligned}$$

(5 points)

(e) For all $\lambda \in \mathbb{K} \setminus \{0, f(z)\}$ the linear operator

$$S_\lambda = -\frac{1}{\lambda} - \frac{1}{\lambda(\lambda - f(z))} T$$

is well-defined and bounded since it is a linear combination of two bounded operators (namely the identity and T). Note that

$$(T - \lambda) S_\lambda x = T S_\lambda x - \lambda x = f(S_\lambda x) z + x + \frac{1}{\lambda - f(z)} T x$$

and that

$$\begin{aligned} f(S_\lambda x) z &= f\Big(-\frac{1}{\lambda} x - \frac{1}{\lambda(\lambda - f(z))} T x \Big) z \\ &= f\Big(\frac{-(\lambda - f(z)) x - f(x) z}{\lambda(\lambda - f(z))} \Big) z \\ &= -\frac{1}{\lambda(\lambda - f(z))} \big((\lambda - f(z)) f(x) + f(x) f(z) \big) z \\ &= -\frac{1}{\lambda - f(z)} T x \end{aligned}$$

which shows that $(T - \lambda) S_\lambda = I$. Likewise, it can be shown that $S_\lambda (T - \lambda) = I$. This proves that $(T - \lambda)^{-1} = S_\lambda \in B(X)$ for all $\lambda \in \mathbb{K} \setminus \{0, f(z)\}$, which implies that $\mathbb{K} \setminus \{0, f(z)\} \subset \rho(T)$.
(3 points)

On the other hand we already know that $\{0, f(z)\} \subset \sigma(T)$ so we actually have $\rho(T) = \mathbb{K} \setminus \{0, f(z)\}$ and $\sigma(T) = \{0, f(z)\}$.
(2 points)

Solution of problem 3 (5 + 3 + 7 + 5 = 20 points)

(a) Let X be a Banach space and let Y be a normed linear space. Let $F \subset B(X,Y)$ and assume that the set

$$M = \{x \in X \,:\, \sup_{T \in F} \|Tx\| < \infty\}$$

is nonmeager. Then the elements $T \in F$ are uniformly bounded:

$$\sup_{T \in F} \|T\| < \infty.$$

(5 points)

(b) Clearly, T_y is linear since Q is linear in the first component for fixed y. For each $x \in X$ we have the following inequality

$$|T_y(x)| = \frac{|Q(x,y)|}{\|y\|} \leq \frac{N_y}{\|y\|}\|x\|.$$

This implies that T_y is a bounded linear operator from X to $\mathbb{K}$.
(3 points)

(c) For each fixed $y \in Y$ with $y \neq 0$ we have the inequality

$$|T_y(x)| = \frac{|Q(x,y)|}{\|y\|} \leq \frac{M_x\|y\|}{\|y\|} = M_x \quad \text{for each } x \in X.$$

(3 points)

Taking the supremum over $y \in Y \setminus \{0\}$ gives

$$\sup_{y \neq 0} |T_y(x)| < \infty \quad \text{for each } x \in X.$$

(2 points)

Since X is a Banach space we can apply the uniform boundedness principle to the set $F = \{T_y \,:\, y \in Y \setminus \{0\}\} \subset B(X, \mathbb{K})$. It follows that

$$K := \sup_{y \neq 0} \|T_y\| < \infty,$$

(2 points)

(d) Finally, for each $y \neq 0$ and $x \in X$ we have

$$\frac{|Q(x,y)|}{\|y\|} = \|T_y(x)\| \leq \|T_y\|\|x\| \leq K\|x\|$$

or, equivalently,

$$|Q(x,y)| \leq K\|x\|\|y\|.$$

(4 points)

Since $Q(x,0) = 0$ for all $x \in X$ this inequality also holds for $y = 0$.
(1 point)

Solution of problem 4 (3 + 7 + 4 + 6 = 20 points)

(a) Let X be a normed linear space and $V \subset X$ a linear subspace. For each $f \in V'$ there exists $F \in X'$ such that $F \upharpoonright V = f$ and $\|F\| = \|f\|$.
(3 points)

(b) For $i = 1, \dots, n$ define $f_i : V \to \mathbb{K}$ by setting

$$f_i(c_1 e_1 + \cdots + c_n e_n) = c_i.$$

Clearly, $f_i : V \to \mathbb{K}$ is a linear map. By construction we have $f_i(e_j) = \delta_{ij}$.
(3 points)

On the one hand, we can define the following norm on V:

$$\|c_1 e_1 + \cdots + c_n e_n\|_V = \max\{|c_1|, \dots, |c_n|\}.$$

On the other hand, the norm $\|\cdot\|$ on X is also a norm on V. Since V is finite-dimensional, the norms $\|\cdot\|$ and $\|\cdot\|_V$ are equivalent on V. In particular, there exists a constant $M > 0$ such that $\|v\|_V \leq M\|v\|$ for all $v \in V$. If $v = c_1 e_1 + \cdots + c_n e_n$, then

$$|f_i(v)| = |c_i| \leq \max\{|c_1|, \dots, |c_n|\} = \|v\|_V \leq M\|v\|,$$

which shows that $f_i : V \to \mathbb{K}$ is bounded.
(3 points)

Now apply the Hahn-Banach Theorem to extend the functionals f_i to all of X while preserving their norm.
(1 point)

(c) It is clear that $Pe_j = e_j$ for all $j = 1, \dots, n$. For any $x \in X$ we have

$$P^2 x = \sum_{i=1}^{n} f_i(x) P e_i = \sum_{i=1}^{n} f_i(x) e_i = Px,$$

which shows that $P^2 = P$.
(2 points)

We have

$$\|Px\| \leq \sum_{i=1}^{n} |f_i(x)| \|e_i\| \leq \left(\sum_{i=1}^{n} \|f_i\| \|e_i\| \right) \|x\|,$$

which shows that P is bounded.
(2 points)

(d) (i) $\operatorname{ran} P$ is finite-dimensional and hence closed. Since P is bounded $\ker P$ is also closed.
(2 points)

(ii) If $x \in \ker P \cap \operatorname{ran} P$, then $Px = 0$ and $x = Py$ for some $y \in X$. This implies $x = Py = P^2 y = Px = 0$. Hence, $\ker P \cap \operatorname{ran} P \subset \{0\}$. The reverse inclusion is trivial.
(2 points)

(iii) Let $x \in X$ be arbitrary, then $x = (I - P)x + Px$ and since $P(I - P)x = Px - P^2x = Px - Px = 0$ it follows that $x \in \ker P + \operatorname{ran} P$. Hence, $X \subset \ker P + \operatorname{ran} P$. The reverse inclusion is trivial.
(2 points)